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Another view on martingale central limit theorems

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Based on the martingale version of the Skorokhod embedding Heyde and Brown (1970) established a bound on the rate of convergence in the central limit theorem (CLT) for discrete time martingales having finite moments of order $2+2\delta$ with $0<\delta\leq 1$. An extension for all $\delta>0$ was proved in Haeusler (1988). This paper presents a rather quick access based solely on truncation, optional stopping, and prolongation techniques for martingale difference arrays $(\xi_{ni}, \mathscr{F}_{ni}, 1\leq i\leq i(n), n\in\mathbb{N})$ to obtain other upper bounds for $\sup_{x\in\mathbb{R}}|\mathbb{P}(\sum_{i=1}^{i(n)}\xi_{ni}\leq x)-\Phi(x)|$ (Φ being the standard normal d.f.) yielding weak sufficient conditions for the asymptotic normality of $\sum_{i=1}^{i(n)}\xi_{ni}$. It is shown that our approach also yields two types of martingale central limit theorems with random norming.

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CLT's for martingales * Lindeberg–Lévy method * rates of convergence * sufficient conditions for asymptotic normality * CLT's with random norming

1. Introduction

For each integer $n\in\mathbb{N}=\{1,2,\ldots\}$, let the real-valued random variables $\xi_1,\ldots,\xi_n$ form a martingale difference sequence (m.d.s. for short) w.r.t. the σ-fields $\mathscr{F}_0\subset\mathscr{F}_1\subset\cdots\subset\mathscr{F}_n$, i.e., suppose that ξ_i is measurable w.r.t. $\mathscr{F}_i$ with $\mathbb{E}(\xi_i|\mathscr{F}_{i-1})=0$ a.s. for $i=1,\ldots,n$. Given a subsequence $(i(n))_{n\in\mathbb{N}}$ of $\mathbb{N}$ with $i(n)\to\infty$ as $n\to\infty$, an array $\xi_{n1},\ldots,\xi_{ni(n)}$, $n\in\mathbb{N}$, of real-valued random variables is called a martingale difference array (m.d.a.) if for each n $\xi_{n1},\ldots,\xi_{ni(n)}$ form a m.d.s. w.r.t. given σ-fields $\mathscr{F}_{n0}\subset\mathscr{F}_{n1}\subset\cdots\subset\mathscr{F}_{ni(n)}$.

Under various sets of assumptions, many authors have shown asymptotic normality of $\sum_{i=1}^{i(n)}\xi_{ni}$, that is,

$$\sum_{i=1}^{i(n)}\xi_{ni}\xrightarrow{\mathscr{L}}\mathscr{N}(0,1), \tag{1.1}$$

or, equivalently,

$$D_{ni(n)}\equiv\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\sum_{i=1}^{i(n)}\xi_{ni}\leq x\right)-\Phi(x)\right|\to 0,\quad\text{as } n\to\infty. \tag{1.2}$$

Here $\mathcal{N}(0,1)$ denotes a standard normal random variable and Φ its distribution function. According to one of the basic results of martingale central limit theory, the 'conditional Lindeberg condition'

$$\sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 I(|\xi_{ni}| > \varepsilon) \mid \mathcal{F}_{n,i-1}) \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \to \infty \text{ for each } \varepsilon > 0, \tag{1.3}$$

and the 'conditional normalizing condition'

$$\sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathcal{F}_{n,i-1}) \xrightarrow{\mathbb{P}} 1, \quad \text{as } n \to \infty, \tag{1.4}$$

together imply (1.1). Obviously, (1.3) is satisfied in particular if for some $\delta > 0$ the 'conditional Liapounov condition of order $2+2\delta$' holds, i.e.,

$$\sum_{i=1}^{i(n)} \mathbb{E}(|\xi_{ni}|^{2+2\delta} \mid \mathcal{F}_{n,i-1}) \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \to \infty. \tag{1.5}$$

Even more, there are results on the rate at which $D_{ni(n)}$ converges to zero. For example, as explained by Hall and Heyde (1981), it is desirable to have bounds on $D_{ni(n)}$ under minimally strengthened versions of conditions (1.4) and (1.5), that is, demanding that these conditions hold in an L_p-norm instead of in probability. Clearly then, under such moment assumptions, one can w.l.o.g. consider bounds on

$$D_n \equiv \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\sum_{i=1}^{n} \xi_i \leq x \right) - \Phi(x) \right|,$$

for a fixed m.d.s. $\xi_1, \ldots, \xi_n$ w.r.t. the σ-fields $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$, where the bound is a function of the moment terms

$$L_{n,2\delta} \equiv \sum_{i=1}^{n} \mathbb{E}(|\xi_i|^{2+2\delta})$$

and

$$N_{n,2\delta} \equiv \mathbb{E}\left(\left| \sum_{i=1}^{n} \mathbb{E}(\xi_i^2 \mid \mathcal{F}_{i-1}) - 1 \right|^{1+\delta} \right).$$

For $0 < \delta \leq 1$, Heyde and Brown (1970) showed that there exists a finite constant C_δ depending only on δ such that for each m.d.s. $\xi_1, \ldots, \xi_n$,

$$D_n \leq C_\delta (L_{n,2\delta} + N_{n,2\delta})^{1/(3+2\delta)}. \tag{1.6}$$

Of course, this inequality is nontrivial only if both $L_{n,2\delta}$ and $N_{n,2\delta}$ are finite, and it provides a rate of convergence in law of $\sum_{i=1}^{n} \xi_i$ to $\mathcal{N}(0,1)$ if $L_{n,2\delta} + N_{n,2\delta} \to 0$ as $n \to \infty$.

In their proof Heyde and Brown (1970) applied the martingale version of the Skorokhod embedding. Erickson, Quine and Weber (1979) obtained (1.6) for $0 \leq \delta \leq \frac{1}{2}$ using the classical characteristic function technique. In Haeusler (1988) a version of Bolthausen's (1982) iterative method is developed to establish (1.6) for every $\delta > 0$.

A simple proof of (1.6) for $0<\delta\leq\frac{1}{2}$ via the Lindeberg–Lévy-method (cf. Lindeberg, 1922; and Lévy, 1937) is possible along the lines in Haeusler (1985) and (1987) yielding for $\delta=\frac{1}{2}$ a value of C_δ which is smaller than the value given by Erickson, Quine and Weber (1979), thus demonstrating the efficiency of the Lindeberg–Lévy-method compared with the characteristic function technique.

Based on Burkholder's inequality (cf. Theorem 2.10 in Hall and Heyde, 1980), if one considers instead of $N_{n,2\delta}$ the so-called Raikov term

$$R_{n,2\delta}\equiv\mathbb{E}\left(\left|\sum_{i=1}^{n}\xi_i^2-1\right|^{1+\delta}\right),$$

it follows from (1.6) that for each $0<\delta\leq\frac{1}{2}$ there exists $0<C_\delta<\infty$ such that for each m.d.s. $\xi_1,\ldots,\xi_n$,

$$D_n\leq C_\delta[L_{n,2\delta}+R_{n,2\delta}]^{1/(3+2\delta)}. \tag{1.6*}$$

Now, the purpose of this paper is to obtain other bounds for D_n yielding weak sufficient conditions for (1.1) to hold. We will show that there exists a generic finite constant C such that for each m.d.s. $\xi_1,\ldots,\xi_n$ w.r.t. the σ-fields $\mathscr{F}_0\subset\mathscr{F}_1\subset\cdots\subset\mathscr{F}_n$,

$$\begin{aligned}D_n\leq C\Bigg[&\mathbb{P}\left(\left|\sum_{i=1}^{n}\mathbb{E}(\xi_i^2\mid\mathscr{F}_{i-1})-1\right|>\beta^2\right)+\mathbb{P}\left(\max_{1\leq i\leq n}|\xi_i|>\beta\right)\\&+\mathbb{P}\left(\sum_{i=1}^{n}\mathbb{E}\left(\xi_i^2I(|\xi_i|>\beta)\,\middle|\,\mathscr{F}_{i-1}\right)>\beta^2\right)+\beta^{1/4}\Bigg]\\&\text{for each }\beta>0,\end{aligned} \tag{1.7}$$

or

$$\begin{aligned}D_n\leq C\Bigg[&\mathbb{P}\left(\left|\sum_{i=1}^{n}\xi_i^2-1\right|>\beta\right)+\beta^{-1}\mathbb{E}\left(\max_{1\leq i\leq n}|\xi_i|I(|\xi_i|>\beta)\right)+\beta^{1/4}\Bigg]\\&\text{for each }\beta>0,\end{aligned} \tag{1.8}$$

respectively, from which one easily derives, for example, (1.1) under the conditions (1.3) and (1.4), known as Brown's (1971) theorem, or a corresponding result of McLeish (1974), respectively.

Besides using (1.6) for $\delta=\frac{1}{2}$ the proof of (1.7) and (1.8) will be solely based on proper truncation, optional stopping and prolongation techniques for m.d.s. In the same spirit we are able to prove certain analogues of (1.7) and (1.8) yielding sufficient conditions for martingale central limit theorems to hold with random norming, i.e. for results like

$$V_n^{-1}\sum_{i=1}^{i(n)}\xi_{ni}\xrightarrow{\mathscr{L}}\mathcal{N}(0,1)$$

or

$$U_n^{-1}\sum_{i=1}^{i(n)}\xi_{ni}\xrightarrow{\mathscr{L}}\mathcal{N}(0,1)$$

with $V_n^2\equiv\sum_{i=1}^{i(n)}\mathbb{E}(\xi_{ni}^2\mid\mathscr{F}_{n,i-1})$ and $U_n^2\equiv\sum_{i=1}^{i(n)}\xi_{ni}^2$, respectively.

Throughout Sections 2 to 3, the following conventions will be used to simplify the notation. The symbol C always denotes a generic finite absolute constant (whose value may change along the proofs without indicating that), whereas C_δ is always a generic finite constant depending only on δ. Equations, inequalities, etc., between random variables are always assumed to hold almost surely without explicit mention, especially when conditional expectations are involved.

2. Preliminaries

In this section we collect two auxiliary results needed within the following section.

To state the first result we define for a real valued random variable ξ,

$$D(\xi) \equiv \sup_{x \in \mathbb{R}} |\mathbb{P}(\xi \leq x) - \Phi(x)|.$$

Lemma 2.1. *Let ξ and η be arbitrary real-valued random variables defined on a common p-space $(\Omega, \mathcal{F}, \mathbb{P})$; then*

$$D(\eta) \leq D(\xi) + a(2\pi)^{-1/2} + \mathbb{P}(|\xi - \eta| > a) \quad \textit{for each } a \geq 0. \qquad \square$$

The second result generalizes Lemma 1 in Haeusler (1984) to conditional probabilities.

Lemma 2.2. *Let $\xi_1, \ldots, \xi_n$ be a m.d.s. w.r.t. $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$ and let τ be a stopping time (w.r.t. $(\mathcal{F}_i)$); then, for all γ, u, $v > 0$,*

$$\begin{aligned}
&\mathbb{P}\left(\max_{\tau < j \leq n} \left| \sum_{i=\tau+1}^{j} \xi_i \right| \geq \gamma \,\Big|\, \mathcal{F}_\tau \right) \\
&\leq 2\mathbb{P}\left(\max_{\tau < i \leq n} |\xi_i| > u \,\Big|\, \mathcal{F}_\tau \right) + 2\mathbb{P}\left(\sum_{i=\tau+1}^{n} \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}) > v \,\Big|\, \mathcal{F}_\tau \right) \\
&\quad + 2\exp\left[\frac{\gamma}{u} - \left(\frac{\gamma}{u} + \frac{v}{u^2} \right) \ln\left(1 + \frac{\gamma u}{v} \right) \right]. \qquad \square
\end{aligned}$$

3. Main results

As announced in the Introduction, our first aim is now to prove (1.7) and (1.8).

Theorem 3.1. *There exists a generic finite constant C such that for each m.d.s. $\xi_1, \ldots, \xi_n$ w.r.t. $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$,*

$$\begin{aligned}
&\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\sum_{i=1} \xi_i \leq x \right) - \Phi(x) \right| \\
&\leq C \left[\mathbb{P}\left(\left| \sum_{i=1}^{n} \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}) - 1 \right| > \beta^2 \right) + \mathbb{P}\left(\max_{1 \leq i \leq n} |\xi_i| > \beta \right) \right. \\
&\qquad \left. + \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{E}(\xi_i^2 I(|\xi_i| > \beta) | \mathcal{F}_{i-1}) > \beta^2 \right) + \beta^{1/4} \right] \quad \textit{for each } \beta > 0.
\end{aligned}$$

Proof. W.l.o.g. we may assume that $0<\beta<\frac{1}{4}$, say. The proof will proceed as follows: Starting with $(\xi_i, \mathscr{F}_i, 1\le i\le n)$, proper truncation and optional stopping will yield a m.d.s. $(\zeta_i, \mathscr{F}_i, 1\le i\le n)$ from which we will construct, by a suitable prolongation technique, another m.d.s. $(\eta_i, \mathscr{F}_i, 1\le i\le N)$, $N>n$, for which the following two properties hold:

$$|\eta_i|\le 2\beta,\quad 1\le i\le N,\quad\text{and}\quad 1-3\beta^2\le\sum_{i=1}^{N}\mathbb{E}(\eta_i^2\,|\,\mathscr{F}_{i-1})\le 1+\beta^2, \tag{$*$}$$

and

$$D\Big(\sum_{i=1}^{n}\xi_i\Big)\le D\Big(\sum_{i=1}^{N}\eta_i\Big)+C[\beta+T_1+T_2+T_3], \tag{$**$}$$

where T_i, $1\le i\le 3$, denote the first three terms on the right-hand side of the stated inequality in Theorem 3.1. From $(*)$ we obtain that

$$L_{N,1}\equiv\sum_{i=1}^{N}\mathbb{E}(|\eta_i|^3)\le 2\beta\sum_{i=1}^{N}\mathbb{E}(\eta_i^2)=2\beta\mathbb{E}\Big(\sum_{i=1}^{N}\mathbb{E}(\eta_i^2\,|\,\mathscr{F}_{i-1})\Big)\le C_1\beta$$

and

$$N_{N,1}\equiv\mathbb{E}\Big(\Big|\sum_{i=1}^{N}\mathbb{E}(\eta_i^2\,|\,\mathscr{F}_{i-1})-1\Big|^{1+1/2}\Big)\le C_2\beta$$

for suitable finite constants C_i, $i=1,2$, and therefore (1.6) (with $\delta=\frac{1}{2}$) implies that

$$D\Big(\sum_{i=1}^{N}\eta_i\Big)\le C\beta^{1/4},$$

which, together with $(**)$ yields the stated inequality in Theorem 3.1.

To achieve this, set, for $1\le i\le n$,

$$\bar{\xi}_i=\xi_i I(|\xi_i|\le\beta)-\mathbb{E}(\xi_i I(|\xi_i|\le\beta)\,|\,\mathscr{F}_{i-1})$$

and

$$\bar{\bar{\xi}}_i=\xi_i I(|\xi_i|>\beta)-\mathbb{E}(\xi_i I(|\xi_i|>\beta)\,|\,\mathscr{F}_{i-1}),$$

then both $\bar{\xi}_1,\dots,\bar{\xi}_n$ and $\bar{\bar{\xi}}_1,\dots,\bar{\bar{\xi}}_n$ are m.d.s. w.r.t. $\mathscr{F}_0\subset\mathscr{F}_1\subset\cdots\subset\mathscr{F}_n$ with $\bar{\xi}_i+\bar{\bar{\xi}}_i=\xi_i$, and $|\bar{\xi}_i|\le 2\beta$ for each $1\le i\le n$. Furthermore, by Lemma 2.1, for each $a\ge 0$,

$$D\Big(\sum_{i=1}^{n}\xi_i\Big)\le D\Big(\sum_{i=1}^{n}\bar{\xi}_i\Big)+a+\mathbb{P}\Big(\Big|\sum_{i=1}^{n}\bar{\bar{\xi}}_i\Big|>2a\Big),$$

where

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} \bar{\bar{\xi}}_i\right| > 2a\right)$$
$$\leq \mathbb{P}\left(\left|\sum_{i=1}^{n} \xi_i I(|\xi_i|>\beta)\right| > a\right) + \mathbb{P}\left(\left|\sum_{i=1}^{n} \mathbb{E}(\xi_i I(|\xi_i|>\beta)\,|\,\mathscr{F}_{i-1})\right| > a\right)$$
$$\leq \mathbb{P}\left(\max_{1\leq i\leq n} |\xi_i| > \beta\right) + \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{E}(\xi_i^2 I(|\xi_i|>\beta)\,|\,\mathscr{F}_{i-1}) > a\beta\right).$$

Thus, choosing $a=\beta$, we get

$$D\left(\sum_{i=1}^{n} \xi_i\right) \leq D\left(\sum_{i=1}^{n} \bar{\xi}_i\right) + \beta + T_2 + T_3. \tag{3.1}$$

Now, set

$$\tau = \sup\left\{ l \in \{0, \ldots, n\}: \sum_{i=1}^{l} \mathbb{E}(\bar{\xi}_i^2\,|\,\mathscr{F}_{i-1}) \leq 1+\beta^2 \right\}$$

and

$$\zeta_i = \bar{\xi}_i I(\tau \geq i), \quad 1 \leq i \leq n,$$

then τ is a stopping time (w.r.t. $(\mathscr{F}_i)$) and $\zeta_1, \ldots, \zeta_n$ is a m.d.s. w.r.t. $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$. Furthermore, by Lemma 2.1 (with $a=0$),

$$D\left(\sum_{i=1}^{n} \bar{\xi}_i\right) \leq D\left(\sum_{i=1}^{n} \zeta_i\right) + \mathbb{P}\left(\sum_{i=1}^{n} \bar{\xi}_i \neq \sum_{i=1}^{n} \zeta_i\right),$$

where, according to the definition of τ,

$$\sum_{i=1}^{n} \bar{\xi}_i \neq \sum_{i=1}^{n} \zeta_i \;\Rightarrow\; \tau < n \;\Rightarrow\; \sum_{i=1}^{n} \mathbb{E}(\bar{\xi}_i^2\,|\,\mathscr{F}_{i-1}) > 1+\beta^2.$$

Since, on the other hand

$$\mathbb{E}(\bar{\xi}_i^2\,|\,\mathscr{F}_{i-1}) \leq \mathbb{E}(\xi_i^2 I(|\xi_i| \leq \beta)\,|\,\mathscr{F}_{i-1}) \leq \mathbb{E}(\xi_i^2\,|\,\mathscr{F}_{i-1}),$$

it follows that

$$\mathbb{P}\left(\sum_{i=1}^{n} \bar{\xi}_i \neq \sum_{i=1}^{n} \zeta_i\right) \leq \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{E}(\xi_i^2\,|\,\mathscr{F}_{i-1}) > 1+\beta^2\right)$$
$$\leq \mathbb{P}\left(\left|\sum_{i=1}^{n} \mathbb{E}(\xi_i^2\,|\,\mathscr{F}_{i-1}) - 1\right| > \beta^2\right),$$

and therefore

$$D\left(\sum_{i=1}^{n} \bar{\xi}_i\right) \leq D\left(\sum_{i=1}^{n} \zeta_i\right) + T_1. \tag{3.2}$$

Our next step consists in a proper prolongation of $(\zeta_i, \mathscr{F}_i, 1 \leq i \leq n)$ to obtain the desired m.d.s. $(\eta_i, \mathscr{F}_i, 1 \leq i \leq N)$ having the properties $(*)$ and $(**)$ stated above. For this, let ζ_i, $i > n$, be real-valued random variables such that

$$\mathbb{P}(\zeta_i = \beta) = \tfrac{1}{2} = \mathbb{P}(\zeta_i = -\beta),$$

and such that $\mathscr{F}_n, \zeta_{n+1}, \zeta_{n+2}, \ldots$, are independent. Set

$$\mathscr{F}_i = \sigma(\mathscr{F}_n, \zeta_{n+1}, \ldots, \zeta_i) \quad \text{for } i > n$$

and

$$\nu = \inf\left\{ l \in \mathbb{N}\colon \sum_{i=1}^{l} \mathbb{E}(\zeta_i^2 \mid \mathscr{F}_{i-1}) \geq 1 - 3\beta^2 \right\},$$

then ν is a stopping time (w.r.t. $(\mathscr{F}_i)$) with $\nu \leq n + [\beta^{-2}] + 1 \equiv N$ (where $[x]$ denotes the integer part of x). Now set,

$$\eta_i = \zeta_i \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad \eta_i = \zeta_i I(\nu \geq i) \quad \text{for } n+1 \leq i \leq N,$$

then $\eta_1, \ldots, \eta_N$ is a m.d.s. w.r.t. $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_N$ with $|\eta_i| \leq 2\beta$ for all $1 \leq i \leq N$, and $1 - 3\beta^2 \leq \sum_{i=1}^{N} \mathbb{E}(\eta_i^2 \mid \mathscr{F}_{i-1}) \leq 1 + \beta^2$, which proves $(*)$.

It remains to verify $(**)$: Again applying Lemma 2.1 (with $a = 0$), we get

$$D\left(\sum_{i=1}^{n} \zeta_i \right) \leq D\left(\sum_{i=1}^{N} \eta_i \right) + \mathbb{P}\left(\sum_{i=1}^{n} \zeta_i \neq \sum_{i=1}^{N} \eta_i \right),$$

where, according to the definition of ν,

$$\sum_{i=1}^{n} \zeta_i \neq \sum_{i=1}^{N} \eta_i \;\Rightarrow\; \sum_{i=1}^{n} \mathbb{E}(\zeta_i^2 \mid \mathscr{F}_{i-1}) < 1 - 3\beta^2,$$

and where, according to the definition of the ζ_i's and of τ,

$$\sum_{i=1}^{n} \mathbb{E}(\zeta_i^2 \mid \mathscr{F}_{i-1}) = \sum_{i=1}^{\tau} \mathbb{E}(\bar{\xi}_i^2 \mid \mathscr{F}_{i-1})$$

and

$$\left\{ \sum_{i=1}^{\tau} \mathbb{E}(\bar{\xi}_i^2 \mid \mathscr{F}_{i-1}) < 1 - 3\beta^2 \right\} \subset \{\tau = n\}.$$

Thus

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{n} \zeta_i \neq \sum_{i=1}^{N} \eta_i \right) \\
&\quad \leq \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{E}(\zeta_i^2 \mid \mathscr{F}_{i-1}) < 1 - 3\beta^2 \right) \\
&\quad \leq \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{E}(\bar{\xi}_i^2 \mid \mathscr{F}_{i-1}) < 1 - 3\beta^2 \right) \\
&\quad \leq \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{E}(\xi_i^2 \mid \mathscr{F}_{i-1}) < 1 - \beta^2 \right) \\
&\qquad + \mathbb{P}\left(\sum_{i=1}^{n} (\mathbb{E}(\xi_i^2 \mid \mathscr{F}_{i-1}) - \mathbb{E}(\bar{\xi}_i^2 \mid \mathscr{F}_{i-1})) > 2\beta^2 \right) \\
&\quad \leq \mathbb{P}\left(\left| \sum_{i=1}^{n} \mathbb{E}(\xi_i^2 \mid \mathscr{F}_{i-1}) - 1 \right| > \beta^2 \right) + \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{E}(\xi_i^2 I(|\xi_i| > \beta) \mid \mathscr{F}_{i-1}) > \beta^2 \right).
\end{aligned}$$

Therefore we get

$$D\left(\sum_{i=1}^{n} \zeta_i\right) \leq D\left(\sum_{i=1}^{N} \eta_i\right) + T_1 + T_3. \tag{3.3}$$

Now, (**) follows from (3.1)-(3.3). $\square$

With an analogous method of proof using the inequality (1.6*) (with $\delta = \frac{1}{2}$) instead of (1.6) (with $\delta = \frac{1}{2}$) one gets:

Theorem 3.2. *There exists a generic finite constant C such that for each m.d.s.* $\xi_1, \ldots, \xi_n$ *w.r.t.* $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\sum_{i=1}^{n} \xi_i \leq x\right) - \Phi(x) \right|$$
$$\leq C\left[\mathbb{P}\left(\left|\sum_{i=1}^{n} \xi_i^2 - 1\right| > \beta^2\right) + \beta^{-1}\mathbb{E}\left(\max_{1 \leq i \leq n} |\xi_i| I(|\xi_i| > \beta)\right) + \beta^{1/4}\right]$$

for each $\beta > 0$. $\square$

Remarks on a sharpening of Theorem 3.1. Let $(\eta_i, \mathscr{F}_i, 1 \leq i \leq N)$ be a m.d.s. Then, as a consequence of the main theorem in Joos (1991), one obtains the following result:

(+) *If* $|\eta_i| \leq 2\beta$, $1 \leq i \leq N$, *and if we put*

$$N_{N,*} \equiv \left\| \sum_{i=1}^{N} \mathbb{E}(\eta_i^2 | \mathscr{F}_{i-1}) - 1 \right\|_{\infty},$$

then there exists a universal constant $0 < C < \infty$ *such that*

$$D\left(\sum_{i=1}^{N} \eta_i\right) \leq C[\beta \ln(\mathrm{e} + \beta^{-2}) + N_{N,*}^{1/2}].$$

Using (+) instead of (1.6) in the proof of Theorem 3.1, the latter result can be sharpened to:

(++) *Let* K_i, $i = 1, 2$, *be arbitrary positive constants. Then there exists a universal constant* $0 < C(K_1, K_2) < \infty$ (*depending only on* K_1 *and* K_2) *such that for any m.d.s.* $(\xi_i, \mathscr{F}_i, 1 \leq i \leq n)$,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\sum_{i=1}^{n} \xi_i \leq x\right) - \Phi(x) \right|$$
$$\leq C(K_1, K_2)\left[\mathbb{P}\left(\left|\sum_{i=1}^{n} \mathbb{E}(\xi_i^2 | \mathscr{F}_{i-1}) - 1\right| > K_2\beta^2\right) + \mathbb{P}\left(\max_{1 \leq i \leq n} |\xi_i| > \beta\right)\right.$$
$$\left. + \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{E}(\xi_1^2 I(|\xi_i| > \beta) | \mathscr{F}_{i-1}) > K_1\beta^2\right) + \beta \ln(\mathrm{e} + \beta^{-2})\right]$$

for each $\beta > 0$.

Furthermore, concerning the probabilities on the right-hand side of the three events

(a) $\left|\sum_{i=1}^{n} \mathbb{E}(\xi_i^2 | \mathscr{F}_{i-1}) - 1\right| > K_2\beta^2,$

(b) $\max_{1\le i\le n} |\xi_i| > \beta,$

(c) $\sum_{i=1}^{n} \mathbb{E}(\xi_i^2 I(|\xi_i| > \beta) | \mathscr{F}_{i-1}) > K_1\beta^2,$

optimality of (++) can be shown (by examples) in the sense that the exponents of β on the right-hand side in (a)–(c) cannot be reduced e.g. to $\beta^{2-\varepsilon}$, $\beta^{1-\varepsilon}$ and $\beta^{2-\varepsilon}$, respectively for any $\varepsilon > 0$.

Finally, Example 2 in Bolthausen (1982) shows that there exists a martingale difference array $(\xi_{ni}, \mathscr{F}_{ni}, 1 \le i \le n)$ for which one can choose $\beta_n \downarrow 0$ such that the probabilities of the corresponding three events (a)–(c) are zero and such that

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\sum_{i=1}^{n} \xi_{ni} \le x\right) - \Phi(x)\right| \frac{1}{\beta_n \ln(e + \beta_n^{-2})} > 0.$$

As mentioned in the Introduction, we are now going to present certain analogues of Theorem 3.1 and Theorem 3.2, respectively, yielding sufficient conditions on a m.d.a. $(\xi_{ni}, \mathscr{F}_{ni}, 1 \le i \le i(n), n \in \mathbb{N})$ for the validity of

$$V_n^{-1} \sum_{i=1}^{i(n)} \xi_{ni} \xrightarrow{\mathscr{L}} \mathscr{N}(0,1) \quad \text{or} \quad U_n^{-1} \sum_{i=1}^{i(n)} \xi_{ni} \xrightarrow{\mathscr{L}} \mathscr{N}(0,1), \tag{3.4}$$

with

$$V_n^2 \equiv \sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 | \mathscr{F}_{n,i-1}) \quad \text{and} \quad U_n^2 \equiv \sum_{i=1}^{i(n)} \xi_{ni}^2, \quad \text{respectively.}$$

Conditions under which assertions like (3.4) hold can be found, for example, in Hall and Heyde (1980, p. 63), even in the stronger form of mixing convergence (being itself a special form of the so-called stable convergence).

Without using the theory of stable or mixing convergence, we will present here a completely different way to deal with CLT's with random norming. In this connection Lemma 2.2 will be an essential tool.

Theorem 3.3. *There exists a generic finite constant C such that for each m.d.s.* $\xi_1, \ldots, \xi_n$ *w.r.t.* $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$, *and for each real-valued and* $\mathscr{F}_m$*-measurable random variable* ζ_m *(where* $m \in \{0, \ldots, n-1\}$*) for which* $|\xi_i \zeta_m^{-1}|$ *is integrable for each* $i > m$, *the following inequality holds*:

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(V_n^{-1}\sum_{i=1}^n \xi_i\le x\right)-\Phi(x)\right|$$
$$\le C\left[\mathbb{P}\left(\left|\sum_{i=1}^n \xi_i\right|\left|V_n^{-1}-\zeta_m^{-1}\right|>\alpha\right)+\mathbb{P}\left(\left|\zeta_m^{-1}\sum_{i=1}^m \xi_i\right|>\alpha\right)\right.$$
$$+\mathbb{P}\left(\sum_{i=m+1}^n \mathbb{E}(\xi_i^2 I(|\xi_i|>\beta\zeta_m)\mid\mathcal{F}_{i-1})>\beta^2\zeta_m^2\right)$$
$$+\mathbb{P}\left(\left|\zeta_m^{-2}\sum_{i=m+1}^n \mathbb{E}(\xi_i^2\mid\mathcal{F}_{i-1})-1\right|>\beta^2\right)+\beta^{1/4}$$
$$\left.+\mathbb{P}\left(\max_{m+1\le i\le n}|\xi_i|>\beta\zeta_m\right)+\alpha\right]$$

for each $\alpha\ge 0$ *and each* $\beta>0$.

Proof. Set $S_i=\sum_{j=1}^i \xi_j$, $1\le i\le n$; applying Lemma 2.1 twice, we get, for each $\alpha\ge 0$,

$$\begin{aligned}D(V_n^{-1}S_n)&\le D(\zeta_m^{-1}S_n)+\alpha(2\pi)^{-1/2}+\mathbb{P}(|S_n|\,|V_n^{-1}-\zeta_m^{-1}|>\alpha)\\&\le D(\zeta_m^{-1}(S_n-S_m))+C\alpha+\mathbb{P}(|\zeta_m^{-1}S_m|>\alpha)\\&\quad+\mathbb{P}(|S_n|\,|V_n^{-1}-\zeta_m^{-1}|>\alpha),\end{aligned}$$

where the last three terms are appearing on the right-hand side of the inequality stated in the theorem. Concerning the other term $D(\zeta_m^{-1}(S_n-S_m))$, it follows from the assumptions on ζ_m that $(\zeta_m^{-1}\xi_i,\ \mathcal{F}_i,\ m+1\le i\le n)$ is a m.d.s., and therefore by Theorem 3.1,

$$\begin{aligned}&D(\zeta_m^{-1}(S_n-S_m))\\&\le C\left[\mathbb{P}\left(\left|\zeta_m^{-2}\sum_{i=m+1}^n \mathbb{E}(\xi_i^2\mid\mathcal{F}_{i-1})-1\right|>\beta^2\right)+\mathbb{P}\left(\max_{m+1\le i\le n}|\xi_i|>\beta\zeta_m\right)\right.\\&\quad\left.+\mathbb{P}\left(\sum_{i=m+1}^n \mathbb{E}(\xi_i^2\zeta_m^{-2}I(|\xi_i|>\beta\zeta_m)\mid\mathcal{F}_{i-1})>\beta^2\right)+\beta^{1/4}\right]\end{aligned}$$

for each $\beta>0$, thus yielding the remaining terms on the right-hand side of the stated inequality in the theorem. □

As a consequence we obtain:

Theorem 3.4. *Let* $(\xi_{ni},\mathcal{F}_{ni},1\le i\le i(n),n\in\mathbb{N})$ *be a m.d.a. and let* η *be a real-valued random variable such that* $\mathbb{P}(0<\eta^2<\infty)=1$. *Assume*

$$\sum_{i=1}^{i(n)}\mathbb{E}(\xi_{ni}^2 I(|\xi_{ni}|>\varepsilon)\mid\mathcal{F}_{n,i-1})\xrightarrow{\mathbb{P}}0,\quad \textit{as } n\to\infty \textit{ for each } \varepsilon>0, \tag{1.3}$$

and

$$V_n^2 \equiv \sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathscr{F}_{n,i-1}) \xrightarrow{\mathbb{P}} \eta^2, \quad \text{as } n \to \infty. \tag{3.5}$$

In addition, assume that either

(a) $\mathscr{F}_{ni} \subset \mathscr{F}_{n+1,i}$ *for each* $n \in \mathbb{N}$ *and for each* $1 \leqslant i \leqslant i(n)$

or

(b) η *is* $\mathscr{F}_{ni}$*-measurable for each* $n \in \mathbb{N}$ *and for each* $1 \leqslant i \leqslant i(n)$.

Then

$$V_n^{-1} \sum_{i=1}^{i(n)} \zeta_{ni} \xrightarrow{\mathscr{L}} \mathscr{N}(0, 1). \tag{3.6}$$

Proof. Concerning the application of Theorem 3.3 below, we choose $0 < \alpha, \beta < \frac{1}{4}$ arbitrary, but fixed. Let K_i, $i = 1, 2$, be positive finite constants such that $\mathbb{P}(\eta \leqslant K_1) \leqslant \frac{1}{2}\beta$ and $\mathbb{P}(\eta \geqslant K_2) \leqslant \frac{1}{2}\beta$. Set

$$\zeta^2 = \eta^2(K_1 \leqslant \eta \leqslant K_2) + K_1^2 I(\eta < K_1) + K_2^2 I(\eta > K_2).$$

We will show first:

(∗) *There exists a subsequence* $(m(n))_{n \in \mathbb{N}}$ *of* $\mathbb{N}$ *such that* $m(n) \uparrow \infty$, $m(n) \leqslant i(n)$ *for each* $n \in \mathbb{N}$, *and* $\sum_{i=1}^{m(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathscr{F}_{n,i-1}) \xrightarrow{\mathbb{P}} 0$, *as* $n \to \infty$.

Since

$$\max_{1 \leqslant i \leqslant i(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathscr{F}_{n,i-1}) \leqslant \varepsilon^2 + \sum_{i=1}^{i(n)} \mathbb{E}(\xi_i^2 I(|\xi_{ni}| > \varepsilon) \mid \mathscr{F}_{n,i-1})$$

for each $\varepsilon > 0$, it follows that (1.3) implies

$$\max_{1 \leqslant i \leqslant i(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathscr{F}_{n,i-1}) \xrightarrow{\mathbb{P}} 0;$$

therefore there exists a sequence $\varepsilon_n \downarrow 0$ such that

$$\lim_{n \to \infty} \mathbb{P}\left(\max_{1 \leqslant i \leqslant i(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathscr{F}_{n,i-1}) > \varepsilon_n \right) = 0.$$

Set $m(n) = \inf\{i(n), [\varepsilon_n^{-1/2}]\}$; then, for each $\varepsilon > 0$,

$$\mathbb{P}\left(\sum_{i=1}^{m(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathscr{F}_{n,i-1}) > \varepsilon \right)$$
$$\leqslant \mathbb{P}\left(\max_{1 \leqslant i \leqslant m(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathscr{F}_{n,i-1}) > \varepsilon \cdot \varepsilon_n^{1/2} \right) \to 0, \quad \text{as } n \to \infty.$$

This proves (∗).

Next, set $\zeta_n^2 = \mathbb{E}(\zeta^2 \mid \mathscr{F}_{n,m(n)})$; then $K_1^2 \leqslant \zeta_n^2 \leqslant K_2^2$ for each $n \in \mathbb{N}$, and *assuming* (a), $\mathscr{F}_{n,m(n)}$ is monotone increasing in n; furthermore since $m(n) \uparrow \infty$,

$\sigma(\bigcup_{n\in\mathbb{N}}\mathscr{F}_{n,m(n)})=\sigma(\bigcup_{n\in\mathbb{N}}\mathscr{F}_{n,i(n)})\equiv\mathscr{F}_\infty$, and therefore, since ζ^2 is $\mathscr{F}_\infty$-measurable, we obtain (cf. Proposition IV-2.3 in Neveu, 1975) $\zeta_n^2\to\zeta^2$ $\mathbb{P}$-almost surely.

Now, by Theorem 3.3 (with $m=m(n)$),

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(V_n^{-1}\sum_{i=1}^{i(n)}\xi_{ni}\le x\right)-\Phi(x)\right|\\
&\le C\left[\mathbb{P}\left(\left|\sum_{i=1}^{i(n)}\xi_{ni}\right|\left|V_n^{-1}-\zeta_n^{-1}\right|>\alpha\right)+\mathbb{P}\left(\left|\zeta_n^{-1}\sum_{i=1}^{m(n)}\xi_{ni}\right|>\alpha\right)\right.\\
&\quad+\mathbb{P}\left(\sum_{i=m(n)+1}^{i(n)}\mathbb{E}(\xi_{ni}^2I(|\xi_{ni}|>\beta\zeta_n)\,|\,\mathscr{F}_{n,i-1})>\beta^2\zeta_n^2\right)\\
&\quad+\mathbb{P}\left(\left|\zeta_n^{-2}\sum_{i=m(n)+1}^{i(n)}\mathbb{E}(\xi_{ni}^2\,|\,\mathscr{F}_{n,i-1})-1\right|>\beta^2\right)+\beta^{1/4}\\
&\quad\left.+\mathbb{P}\left(\max_{m(n)+1\le i\le i(n)}|\xi_{ni}|>\beta\zeta_n\right)+\alpha\right].
\end{aligned}\tag{3.7}$$

We are going to show that

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(V_n^{-1}\sum_{i=1}^{i(n)}\xi_{ni}\le x\right)-\Phi(x)\right|\\
&\le C[\mathbb{P}(\eta^2\ge\alpha^{-1/2})+\exp\{\alpha^{-1}(1-\ln(1+\alpha^{-1/2}))\}+\beta^{1/4}+\alpha],
\end{aligned}\tag{3.8}$$

from which the assertion of the theorem follows, since α and β can be chosen arbitrarily small.

First, one easily shows (cf. p. 522 in Dvoretzky, 1972) that, (1.3) implies

$$(**)\qquad \max_{1\le i\le i(n)}|\xi_{ni}|\xrightarrow{\mathbb{P}}0,$$

and therefore, since $\zeta_n\ge K_1$,

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbb{P}\left(\max_{m(n)+1\le i\le i(n)}|\xi_{ni}|>\beta\zeta_n\right)\\
&\le\limsup_{n\to\infty}\mathbb{P}\left(\max_{1\le i\le i(n)}|\xi_{ni}|>\beta K_1\right)=0.
\end{aligned}\tag{3.9}$$

Applying Lemma 2.2 (with $\tau\equiv 0$, $\gamma=K_1\alpha$, $u=K_1\alpha^2$ and $v=K_1^2\alpha^{7/2}$) to the second summand on the right-hand side of (3.7) yields

$$\begin{aligned}
\mathbb{P}\left(\left|\zeta_n^{-1}\sum_{i=1}^{m(n)}\xi_{ni}\right|>\alpha\right)&\le\mathbb{P}\left(\left|\sum_{i=1}^{m(n)}\xi_{ni}\right|>K_1\alpha\right)\\
&\le 2\mathbb{P}\left(\max_{1\le i\le m(n)}|\xi_{ni}|>K_1\alpha^2\right)\\
&\quad+2\mathbb{P}\left(\sum_{i=1}^{m(n)}\mathbb{E}(\xi_{ni}^2\,|\,\mathscr{F}_{n,i-1})>K_1\alpha^{7/2}\right)\\
&\quad+2\exp\{\alpha^{-1}(1-\ln(1+\alpha^{-1/2}))\}.
\end{aligned}$$

According to (*) and (**) this implies

$$\limsup_{n\to\infty} \mathbb{P}\left(\left|\zeta_n^{-1}\sum_{i=1}^{m(n)} \xi_{ni}\right| > \alpha\right) \leq 2\exp\{\alpha^{-1}(1-\ln(1+\alpha^{-1/2}))\}. \tag{3.10}$$

Next, since $\zeta_n \geq K_1$, it follows from (1.3) that

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{i=m(n)+1}^{i(n)} \mathbb{E}(\xi_{ni}^2 I(|\xi_{ni}| > \beta\zeta_n) \mid \mathscr{F}_{n,i-1}) > \beta^2\zeta_n^2\right) = 0. \tag{3.11}$$

As to the first summand on the right-hand side of (3.7), we get

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |V_n^{-1} - \zeta_n^{-1}| > \alpha\right) \\
&\quad\leq \mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |V_n^{-1} - \eta^{-1}| > \tfrac{1}{3}\alpha\right) + \mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |\eta^{-1} - \zeta^{-1}| > \tfrac{1}{3}\alpha\right) \\
&\qquad + \mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |\zeta_n^{-1} - \zeta^{-1}| > \tfrac{1}{3}\alpha\right),
\end{aligned}$$

where

$$\mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |\eta^{-1} - \zeta^{-1}| > \tfrac{1}{3}\alpha\right) \leq \mathbb{P}(\eta \neq \zeta) \leq \beta,$$

and

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |V_n^{-1} - \eta^{-1}| > \tfrac{1}{3}\alpha\right) \\
&\quad\leq \mathbb{P}(|V_n^{-1} - \eta^{-1}| > \tfrac{1}{3}\alpha^2) \\
&\qquad + \mathbb{P}\left(\left\{\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |V_n^{-1} - \eta^{-1}| > \tfrac{1}{3}\alpha\right\} \cap \{|V_n^{-1} - \eta^{-1}| \leq \tfrac{1}{3}\alpha^2\}\right) \\
&\quad\leq \mathbb{P}(|V_n^{-1} - \eta^{-1}| > \tfrac{1}{3}\alpha^2) + \mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| > \alpha^{-1}\right),
\end{aligned}$$

where a further application of Lemma 2.2 (with $\tau \equiv 0$, $\gamma = \alpha^{-1}$, $u = 1$, and $v = \alpha^{-1/2}$) yields

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| > \alpha^{-1}\right) \\
&\quad\leq 2\mathbb{P}\left(\max_{1\leq i\leq i(n)} |\xi_{ni}| > 1\right) + 2\mathbb{P}\left(\sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 \mid \mathscr{F}_{n,i-1}) > \alpha^{-1/2}\right) \\
&\qquad + 2\exp\{\alpha^{-1}(1-\ln(1+\alpha^{-1/2}))\};
\end{aligned}$$

as to the other summand, since $V_n \xrightarrow{\mathbb{P}} \eta$, $\lim_{n\to\infty} \mathbb{P}(|V_n^{-1} - \eta^{-1}| > \frac{1}{3}\alpha^2) = 0$, and, concerning the first two summands in the last inequality, we get

$$\limsup_{n\to\infty} 2\mathbb{P}\left(\sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 | \mathcal{F}_{n,i-1}) > \alpha^{-1/2}\right) \leq 2\mathbb{P}(\eta^2 \geq \alpha^{1/2}),$$

and, by (**),

$$\lim_{n\to\infty} 2\mathbb{P}\left(\max_{1\leq i\leq i(n)} |\xi_{ni}| > 1\right) = 0.$$

Analogously, one gets

$$\mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |\zeta_n^{-1} - \zeta^{-1}| > \tfrac{1}{3}\alpha\right)$$
$$\leq \mathbb{P}(|\zeta_n^{-1} - \zeta^{-1}| > \tfrac{1}{3}\alpha^2) + \mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| > \alpha^{-1}\right),$$

with the same upper estimate for the second summand as before, and where $\lim_{n\to\infty} \mathbb{P}(|\zeta_n^{-1} - \zeta^{-1}| > \frac{1}{3}\alpha^2) = 0$, since $\zeta_n \to \zeta$ $\mathbb{P}$-almost surely as shown above.

Summarizing we thus obtain

$$\limsup_{n\to\infty} \mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \xi_{ni}\right| |V_n^{-1} - \zeta_n^{-1}| > \alpha\right)$$
$$\leq 4\mathbb{P}(\eta^2 \geq \alpha^{-1/2}) + \beta + 4\exp\{\alpha^{-1}(1 - \ln(1 + \alpha^{-1/2}))\}. \tag{3.12}$$

Finally, concerning the fourth summand on the right-hand side of (3.7),

$$\mathbb{P}\left(\left|\zeta_n^{-2} \sum_{i=m(n)+1}^{i(n)} \mathbb{E}(\xi_{ni}^2 | \mathcal{F}_{n,i-1}) - 1\right| > \beta^2\right)$$
$$\leq \mathbb{P}\left(\left|\sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 | \mathcal{F}_{n,i-1}) - \eta^2\right| > \tfrac{1}{4}K_1^2\beta^2\right)$$
$$+ \mathbb{P}\left(\sum_{i=1}^{m(n)} \mathbb{E}(\xi_{ni}^2 | \mathcal{F}_{n,i-1}) > \tfrac{1}{4}K_1^2\beta^2\right)$$
$$+ \mathbb{P}(|\eta^2 - \zeta^2| > \tfrac{1}{4}K_1^2\beta^2) + \mathbb{P}(|\zeta_n^2 - \zeta^2| > \tfrac{1}{4}K_1^2\beta^2),$$

where $\mathbb{P}(|\eta^2 - \zeta^2| > \frac{1}{4}K_1^2\beta^2) \leq \mathbb{P}(\eta \neq \zeta) \leq \beta$ and, by (3.5) and (*), and since $\zeta_n \to \zeta$ $\mathbb{P}$-a.s., all the other summands are vanishing as $n \to \infty$; thus

$$\limsup_{n\to\infty} \mathbb{P}\left(\left|\zeta_n^{-2} \sum_{i=m(n)+1}^{i(n)} \mathbb{E}(\xi_{ni}^2 | \mathcal{F}_{n,i-1}) - 1\right| > \beta^2\right) \leq \beta \leq \beta^{1/4}. \tag{3.13}$$

Now, inserting (3.9)-(3.13) into (3.7) yields (3.8) proving the theorem under the condition (a).

Next, *assuming* (b), with η also ζ is $\mathscr{F}_{ni}$-measurable for each $n \in \mathbb{N}$ and for each $1 \leq i \leq i(n)$, and therefore by Theorem 3.3 (with $m = 0$) we get

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(V_n^{-1} \sum_{i=1}^{i(n)} \xi_{ni} \leq x \right) - \Phi(x) \right|$$

$$\leq C \Bigg[\mathbb{P}\left(\left| \sum_{i=1}^{i(n)} \xi_{ni} \right| |V_n^{-1} - \zeta^{-1}| > \alpha \right)$$

$$+ \mathbb{P}\left(\sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 I(|\xi_{ni}| > \beta\zeta) \,|\, \mathscr{F}_{n,i-1}) > \beta^2\zeta^2 \right)$$

$$+ \mathbb{P}\left(\left| \zeta^{-2} \sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 \,|\, \mathscr{F}_{n,i-1}) - 1 \right| > \beta^2 \right) + \beta^{1/4}$$

$$+ \mathbb{P}\left(\max_{1\leq i\leq i(n)} |\xi_{ni}| > \beta\zeta \right) + \alpha \Bigg]. \tag{3.14}$$

Again it suffices to prove (3.8). First, in the same way as deriving (3.11) and (3.9), it follows that both the second and fifth summand on the right-hand side of (3.14) are vanishing as $n \to \infty$.

Concerning the first summand, an analogous proof to the proof of (3.12) yields

$$\limsup_{n\to\infty} \mathbb{P}\left(\left| \sum_{i=1}^{i(n)} \xi_{ni} \right| |V_n^{-1} - \zeta^{-1}| > \alpha \right)$$

$$\leq 2\mathbb{P}(\eta^2 \geq \alpha^{-1/2}) + \beta + 2\exp\{\alpha^{-1}(1 - \ln(1 + \alpha^{-1/2}))\}. \tag{3.15}$$

Finally,

$$\mathbb{P}\left(\left| \zeta^{-2} \sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 \,|\, \mathscr{F}_{n,i-1}) - 1 \right| > \beta^2 \right)$$

$$\leq \mathbb{P}(\eta \neq \zeta) + \mathbb{P}\left(\left| \sum_{i=1}^{i(n)} \mathbb{E}(\xi_{ni}^2 \,|\, \mathscr{F}_{n,i-1}) - \eta^2 \right| > K_1^2\beta^2 \right),$$

and thus, summarizing we get (3.8). This proves the theorem. □

Our final results are concerned with CLT's under the random norming $U_n^2 \equiv \sum_{i=1}^{i(n)} \xi_{ni}^2$ instead of V_n. Using Theorem 3.2 instead of Theorem 3.1, one obtains the following analogue of Theorem 3.3:

Theorem 3.5. *There exists a generic finite constant C such that for each m.d.s. $\xi_1, \ldots, \xi_n$ w.r.t. $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$, and for each real-valued and $\mathscr{F}_m$-measurable random variable ζ_m (where $m \in \{0, \ldots, n-1\}$) for which $|\xi_i\zeta_m^{-1}|$ is integrable for each $i > m$, the following* *inequality holds*:

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(U_n^{-1}\sum_{i=1}^{n}\xi_i\le x\right)-\Phi(x)\right|$$
$$\le C\left[\mathbb{P}\left(\left|\sum_{i=1}^{n}\xi_i\right|\left|U_n^{-1}-\zeta_m^{-1}\right|>\alpha\right)+\mathbb{P}\left(\left|\zeta_m^{-1}\sum_{i=1}^{m}\xi_i\right|>\alpha\right)\right.$$
$$+\beta^{-1}\mathbb{E}\left(\max_{m+1\le i\le n}|\zeta_m^{-1}\xi_i|I(|\zeta_m^{-1}\xi_i|>\beta)\right)$$
$$\left.+\mathbb{P}\left(\left|\zeta_m^{-2}\sum_{i=m+1}^{n}\xi_i^2-1\right|>\beta^2\right)+\beta^{1/4}+\alpha\right]$$

for each $\alpha\ge 0$ *and each* $\beta>0$. □

As a consequence one obtains (with an analogous method of proof to that for proving Theorem 3.4) the following result:

Theorem 3.6. *Let* $(\xi_{ni},\mathcal{F}_{ni},1\le i\le i(n),\ n\in\mathbb{N})$ *be a m.d.a. and let* η *be a real-valued random variable such that* $\mathbb{P}(0<\eta^2<\infty)=1$. *Assume*

$$\mathbb{E}\left(\max_{1\le i\le i(n)}|\xi_{ni}|\right)\to 0,\quad \text{as } n\to\infty, \tag{3.16}$$

and

$$U_n^2\equiv\sum_{i=1}^{i(n)}\xi_{ni}^2\xrightarrow{\mathbb{P}}\eta^2,\quad \text{as } n\to\infty. \tag{3.17}$$

In addition, assume that either (a) *or* (b) *of Theorem* 3.4 *hold. Then*

$$U_n^{-1}\sum_{i=1}^{i(n)}\xi_{ni}\xrightarrow{\mathcal{L}}\mathcal{N}(0,1).\qquad \square \tag{3.18}$$

Remarks. Example 1 on p. 65 in Hall and Heyde (1980) shows that (3.5) cannot be weakened to $V_n^2\xrightarrow{\mathcal{L}}\eta^2$. Concerning statistical applications, the relevance of (3.6) and (3.18) lies in the fact that η, being unknown in general, does not appear in the limiting distribution of $V_n^{-1}\sum_{i=1}^{i(n)}\xi_{ni}$ and $U_n^{-1}\sum_{i=1}^{i(n)}\xi_{ni}$, respectively.

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